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ESTIMATES FOR CHARACTER SUMS IN NUMBER FIELDS

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ABSTRACT

We obtain asymptotic formulae for the moments of coefficients of Artin–Weil L-functions (defined by equation (1)). As a by-product of our investigations, we strengthen and generalise the results of several authors cited in the references.

Introduction

Let ρ_j, $1 \leqq j \leqq r$, be a (continuous) complex finite dimensional representation of the Weil group of an algebraic number field k of finite degree over the rationals and let

$$L(s, \rho_j) = \sum_{n} a(n, \rho_j) | n |^{-s}, \qquad 1 \leqq j \leqq r,$$

be the Artin–Weil L-function associated to ρ_j. Here n ranges over the integral ideals of k and $| n | := N_{k/\mathbf{Q}} n$. Let

$$a(n, \boldsymbol{\rho}) = \prod_{j=1}^{r} a(n, \rho_j) \tag{1}$$

and let

$$A(x, \boldsymbol{\rho}) = \sum_{|n| < x} a(n, \boldsymbol{\rho}).$$

The purpose of this article is to give an asymptotic estimate for $A(x, \boldsymbol{\rho})$ as $x \to \infty$ with effective numerical constants in the error term. In the absence of

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Artin conjecture the error term will depend, of course, on the location of exceptional Siegel zeros. As a by-product of our investigations we obtain estimates generalising results discussed by several authors [25]; [6]; [5]; [18]; [12, Appendix]; [17], [22] (cf. also [1], [21], [4]).

Notations and conventions

As usual, **Q**, **R**, **C**, **Z** denote the fields of rational, real and complex numbers, and the ring of rational integers, respectively; $\mathbf{R}_+$ stands for the multiplicative group of positive real numbers. Let F be a number field of degree $n(F) = [F:\mathbf{Q}]$ over **Q**. We denote by $I(F)$ and $C(F)$ the idèle group and the idèle-class group of F, respectively. Embed $\mathbf{R}_+$ diagonally in $I(F)$ and write

$$C(F) \cong C_1(F) \times \mathbf{R}_+, \tag{2}$$

where $C_1(F)$ is the subgroup of $C(F)$ consisting of idèle-classes having unit volume. Let S, S_1, S_2 be the sets of all places, all real places and all the complex places of F, respectively; let F_p denote the completion of F at p in S and let $r_j = \operatorname{card} S_j$, $j = 1, 2$, so that $n(F) = r_1 + 2r_2$. Let $\operatorname{gr}(F)$ be the group of characters of $C_1(F)$. Any χ in $\operatorname{gr}(F)$ may be regarded as a character of $I(F)$ trivial on $\mathbf{R}_+$ and on the subgroup of principal idèles; write

$$\chi = \prod_{p \in S} \chi_p,$$

where χ_p is a (continuous) character of F_p^* for each p, and let

$$\chi_p(\alpha) = |\alpha|^{it_p(\chi)} \left(\frac{\alpha}{|\alpha|}\right)^{a_p(\chi)}, \qquad \alpha \in F_p^* \tag{3}$$

for $p \in S_1 \cup S_2$ with $t_p(\alpha) \in \mathbf{R}$, $a_p(\chi) \in \mathbf{Z}$ and, moreover, $a_p(\chi) \in \{0, 1\}$ when $p \in S_1$. Let

$$a(\chi) = \prod_{p \in S_1} (2 + |t_p(\chi)|)^{1/2} \prod_{p \in S_2} \left(2 + \frac{|t_p(\chi)| + |a_p(\chi)|}{2}\right). \tag{4}$$

Let D and $\mathscr{F}(\chi)$ denote the discriminant of F and the conductor of χ, respectively, and let

$$b(\chi) = \sqrt{|D| N_{F/\mathbf{Q}} \mathscr{F}(\chi)}.$$

Given a finite normal field extension $E \mid F$, one denotes by $G(E \mid F)$ and $W(E \mid F)$ its Galois and Weil groups, respectively. It follows from the definition of a relative Weil group, [26], [23], that parallel to (2) we have a decomposition

$$W(E \mid F) \cong W_1(E \mid F) \times \mathbf{R}_+, \tag{5}$$

where $W_1(E \mid F)$ is a compact group obtained as an extension of $G(E \mid F)$ by $C_1(E)$. The (absolute) Weil group $W(F)$ of F is defined as the projective limit of $W(E \mid F)$, where E varies over finite normal extensions of F (cf. [23]). It follows from the definitions that any continuous representation

$$\rho : W(F) \to \mathrm{GL}(d, C) \tag{6}$$

factors through $W(E \mid F)$ for some $E \mid F$; if, moreover, $\mathbf{R}_+ \subset \operatorname{Ker} \rho$, so that ρ factors through $W_1(E \mid F)$, we say that ρ is normalized. A one-dimensional normalised representation of $W(F)$ may be identified with a grossencharacter in $\mathrm{gr}(F)$. We recall that the L-function associated to ρ is defined by an Euler product:

$$L(s, \rho) = \prod_{p \in S_0} \det(I - \rho(\sigma_p) |p|^{-s})^{-1}, \tag{7}$$

where $S_0 = S \setminus (S_1 \cup S_2)$ is the set of finite places, or prime divisors of F; to define $\rho(\sigma_p)$ we fix an element τ_p in the Frobenius class σ_p and denote by $\rho(\sigma_p)$ the restriction of the operator $\rho(\tau_p)$ to the subspace of the representation space consisting of the vectors fixed by the elements of the inertia group at p. If $\chi = \operatorname{tr} \rho$ is the character of ρ we write, for brevity,

$$\chi(p) = \operatorname{tr} \rho(\sigma_p), \qquad p \in S_0. \tag{8}$$

If ρ is normalised and factors through $W(E \mid F)$, then there are number fields E_j, $1 \leqq j \leqq \nu$, such that $F \subseteq E_j \subseteq E$ for each j, and grossencharacters ψ_j, in $\mathrm{gr}(E_j)$, satisfying the condition

$$L(s, \rho) = \prod_{j=1}^{\nu} L(s, \psi_j) e_j, \qquad e_j = \pm 1, \quad 1 \leqq j \leqq \nu, \tag{9}$$

where $L(s, \psi_j)$ is the Hecke L-function associated with ψ_j. In notations of (9), let

$$a(\chi) = \prod_{j=1}^{\nu} a(\psi_j), \quad b(\chi) = \prod_{j=1}^{\nu} b(\psi_j), \quad \chi := \operatorname{tr} \rho. \tag{10}$$

For an integral divisor n of F and $\psi \in \mathrm{gr}(F)$ the value $\psi(n)$ is defined in a usual way (see, e.g., [3, §9]); in particular, $\psi(\mathfrak{n})=0$ when $(\mathfrak{n}, \mathscr{F}(\psi)) \neq 1$. We write, for brevity,

$$|\mathfrak{n}| := N_{F/\mathbf{Q}}\mathfrak{n}.$$

The constants implied by the O-symbols are effectively computable non-negative real numbers; other numerical effectively computable constants are denoted by $c_1, c_2, \ldots$. We denote by $\zeta(s)$ and $\zeta_F(s)$ the Riemann zeta-function and the Dedekind zeta-function of F, respectively (so that $\zeta(s)=\zeta_{\mathbf{Q}}(s)$).

§1. Statement of the main results

Let k be a number field of degree $n = n(k)$ over $\mathbf{Q}$. A representation

$$\rho : W(k) \to \mathrm{GL}(d, \mathbf{C})$$

is said to be of AW type if the function

$$s \mapsto L(s, \rho)$$

is holomorphic in $\mathbf{C} \setminus \{1\}$. We fix a decomposition (9) and let

$$m = \sum_{j=1}^{\nu} n(E_j). \tag{11}$$

Define the coefficients $a(\mathfrak{n}, \chi)$, $\chi := \operatorname{tr} \rho$, by

$$L(s,\rho) = \sum_{\mathfrak{n}} a(\mathfrak{n}, \chi) |\mathfrak{n}|^{-s}, \qquad \operatorname{Re} s > 1,$$

where $\mathfrak{n}$ ranges over all the integral divisors of k.

THEOREM 1. *Suppose that ρ is of AW type and let l denote the multiplicity of the identical representation in ρ.*

There is a polynomial $P_\rho(t)$ of degree $l-1$ when $l>0$ and equal to zero when $l=0$ such that, for $\varepsilon > 0$, $x \geqq 9$,

$$\sum_{|\mathfrak{n}|<x} a(\mathfrak{n}, \chi) = xP_\rho(\log x) + O(C_1(\varepsilon)(a(\chi)b(\chi))^2(\log x)^{2nd}x^{1-1/(2+m)+\varepsilon}), \tag{12}$$

where $C_1(\varepsilon)$ is a positive valued exactly computable function of ε.

Let

$$\rho_j : W(k) \to \mathrm{GL}(d_j, \mathbf{C}), \qquad 1 \leqq j \leqq r,$$

be a representation of degree d_j, let

$$\rho = \rho_1 \otimes \cdots \otimes \rho_r,$$

and let

$$d = \prod_{j=1}^{r} d_j.$$

We say that $\boldsymbol{\rho}$ is of AW type if ρ is of AW type.

THEOREM 2. *Suppose that $\boldsymbol{\rho}$ is of AW type and let l be the multiplicity of the identical representation in ρ. There is a polynomial $P_{\boldsymbol{\rho}}(t)$ of degree $l-1$ when $l>0$, equal to zero when $l=0$ and such that, for $\varepsilon>0$, $x \geqq 9$,*

$$A(x, \boldsymbol{\rho}) = xP_{\boldsymbol{\rho}}(\log x) + O(C_2(\varepsilon)(a(\chi)b(\chi))x^{1-2/(4+m)+\varepsilon}), \tag{13}$$

where $C_2(\varepsilon)$ may depend on n and d (being as $C_1(\varepsilon)$ a positive valued effectively computable function of ε).

Making no assumption about ρ we can assure only a much weaker estimate for the error term in (13).

THEOREM 3. *There are α_j, β_j, γ_j, $1 \leqq j \leqq \nu$, and c_1, c_2 such that, in notations of Theorem 2,*

$$A(x, \boldsymbol{\rho}) = xP_{\boldsymbol{\rho}}(\log x) + \sum_{j=1}^{\nu} x^{\alpha_j}(\log x)^{\beta_j}\gamma_j + R(\boldsymbol{\rho}, x), \tag{14}$$

and $\alpha_j < 1$ for each j, $c_1 > 0$,

$$R(\boldsymbol{\rho}, x) = O(x \exp(-c_1 m^{-1}\sqrt{\log x})(a(\chi)b(\chi))^{c_2} C_3(d, k, m)) \tag{15}$$

with an exactly computable $C_3(d, k, m)$.

REMARK 1. The constants α_j, β_j, γ_j depend on the location of Siegel zeros of $L(s, \psi_j)$, $1 \leqq j \leqq \nu$.

THEOREM 4. *Let, in notations of Theorem 1, $l = g(\chi)$. We have*

$$\sum_{|p|<x} \chi(p) = g(\chi) \int_2^x \frac{du}{\log u} + O\left(\sum_{j=1}^{\nu} x^{\alpha_j}\right) + R_1(\chi, x), \tag{16}$$

where α_j, $1 \leqq j \leqq \nu$, denotes the possible exceptional zero of $L(s, \psi_j)$ and p ranges over the prime ideals of k; moreover,

$$R_1(\chi, x) = O(m\sqrt{x}) + O\left(x \sum_{j=1}^{\nu} \exp\left(-c_3 \frac{\log x}{\log(a(\psi_j)b(\psi_j)) + \sqrt{n(E_j)}\log x}\right)\right)$$
(17)

for some $c_3 > 0$.

Let $k_j \mid k$ be a finite extension of degree $d_j = [k_j : k]$ and let $\chi_j \in \mathrm{gr}(k_j)$, $1 \leqq j \leqq r$; let

$$\rho_j = \mathrm{Ind}_{W(k_j)}^{W(k)}(\chi_j)$$

be the representation of $W(k)$ induced by χ_j and let

$$\rho = \rho_1 \otimes \cdots \otimes \rho_r, \qquad \chi := \mathrm{tr}\, \rho.$$

Let

$$J(k) = \{\mathfrak{a} \mid \mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r), N_{k_1/k}\mathfrak{a}_1 = \cdots = N_{k_r/k}\mathfrak{a}_r\},$$

where $\mathfrak{a}_j$ varies over integral ideals in k_j, $1 \leqq j \leqq r$, and let

$$(18) \qquad \chi(\mathfrak{a}) = \prod_{j=1}^{\nu} \chi_j(\mathfrak{a}_j), \quad |\mathfrak{a}| = N_{k_1/\mathbf{Q}}\mathfrak{a}_1 = |\mathfrak{a}_j|, \qquad 1 \leqq j \leqq r,$$

for $\mathfrak{a} \in J(k)$.

THEOREM 5. *In notations of Theorem* 2, *the following estimate holds*:

$$(19) \qquad \sum_{|\mathfrak{a}|<x} \chi(\mathfrak{a}) = xP_{\rho}(\log x) + O(C_2(\varepsilon)(a(\chi)b(\chi))x^{1-2/(4+m)+\varepsilon}),$$

where $\mathfrak{a}$ *ranges over* $J(\mathbf{k})$.

REMARK 2. The polynomial $P_{\rho}(t)$ in (19) is exactly computable and in the course of the proof detailed information on its shape is given.

Let

$$J_0(\mathbf{k}) = \{\mathfrak{p} \mid \mathfrak{p} \in J_0(\mathbf{k}), \mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_r), \mathfrak{p}_j \in S_0(k_j), 1 \leqq j \leqq r\},$$

where $S_0(k_j)$ denotes the set of prime divisors in k_j.

THEOREM 6. *In notations of Theorem* 4, *we have*

$$(20) \qquad \sum_{|\mathfrak{p}|<x} \chi(\mathfrak{p}) = \sum_{|p|<x} \chi(p) + O(d\sqrt{x} + C_4(\chi))$$

with a constant $C_4(\chi)$ *exactly expressible in terms of* χ.

Here $\mathfrak{p}$ ranges over the elements of $J_0(\mathbf{k})$, while p varies over prime divisors of k.

We give also the following conditional result.

THEOREM 7. *Suppose that, in the above notations,*

$$L(s, \psi_j) \neq 0 \quad \text{for } \operatorname{Re} s > \tfrac{1}{2}, \qquad 1 \leqq j \leqq v; \tag{21}$$

then the following estimates hold:

$$A(x, \mathfrak{p}) = xP_{\mathfrak{p}}(\log x) + O(C_5(\varepsilon)x^{1/2+\varepsilon}(a(\chi)b(\chi))^{\varepsilon}C_6(d, m)) \tag{22}$$

and the estimate of the same shape for $\sum_{|\mathfrak{a}|<x} \boldsymbol{\chi}(\mathfrak{a})$:

$$\sum_{|p|<x} \chi(p) = g(\chi) \int_2^x \frac{du}{\log u} + O(x^{1/2}(m \log x + \log(a(\chi)b(\chi)))). \tag{23}$$

Here $C_5(\varepsilon)$ is an exactly computable function of ε which may depend on m and d.

The proof of these results follows the pattern of classical analytic number theory as developed in [8], [9] (cf. also [19]); therefore some of the details will be omitted (cf., however, [16]).

§2. Proof of Theorem 1 and Theorem 2

We recall briefly the properties of the scalar product of Artin–Weil L-functions defined by the following Dirichlet series:

$$L(s, \mathfrak{p}) = \sum_n a(n, \mathfrak{p}) |n|^{-s}, \qquad \operatorname{Re} s > 1, \tag{24}$$

where n varies over the integral ideals of k.

THEOREM 8. *There are a finite set $S_0(\mathfrak{p})$ and a system of polynomials*

$$\Phi_p(t) = 1 + \sum_{m=2}^{d-1} b_m(p)t^m$$

such that $S_0(\mathfrak{p}) \subset S_0$, $l_p(t) = \Pi_{j=1}^d (1 - \alpha_j(p)t)$, $|\alpha_j(p)| \in \{0, 1\}$, and

$$L(s, \mathfrak{p}) = L(s, \rho) \prod_{p \in S_0} \Phi_p(|p|^{-s}) \prod_{p \in S_0(\mathfrak{p})} l_p(|p|^{-s}), \quad \operatorname{Re} s > \tfrac{1}{2}, \tag{25}$$

where $\rho = \rho_1 \otimes \cdots \otimes \rho_r$ and

(26) $$b_m(p) = \sum_{m_1+m_2=m} (-1)^{m_1} \operatorname{tr} \Lambda^{m_1}\left(\bigotimes_{j=1}^{r} \rho_j(\sigma_p)\right) \prod_{j=1}^{r} \operatorname{tr}(S^{m_2}\rho_j(\sigma_p)).$$

Here Λ and S denote the exterior power and the symmetric power of a matrix; m_1, m_2 vary over positive integers.

PROOF. See, for instance, [16, p. 85].

LEMMA 1. *In the half-plane* $\operatorname{Re} s > \frac{1}{2}$ *we have an estimate*

(27) $$|L(s, \mathfrak{p})| \leqq c_4(2\sigma - 1)^{d'} |L(s, \rho)|, \qquad \sigma := \operatorname{Re} s,$$

where $d' = -d^{d+3}n$; $n := [k : Q]$.

PROOF. It follows from (25) that

(28) $$|L(s, \mathfrak{p})| \leqq |L(s, \rho)| \prod_{p \in S_0} |\Phi_p(|p|^{-s})|, \qquad \operatorname{Re} s > \tfrac{1}{2}.$$

By (26), $|b_m(p)| \leqq (m+1)d^m$ (since ρ_j is equivalent to a unitary representation). Therefore

$$|\Phi_p(t)| \leqq 1 + d^{d+2}|t|^2 \sum_{j=0}^{d-3} |t|^j \leqq 1 + d^{d+3}|t|^2 \quad \text{for } |t| < 1.$$

Hence

(29) $$\left|\prod_{p \in S_0} \Phi_p(|p|^{-s})\right| \leqq \zeta_k(2\sigma)^{-d'/n} \quad \text{for } \sigma > \tfrac{1}{2}.$$

Inequality (27) follows from (28), (29) and elementary inequalities:

(30) $$\zeta_k(\sigma) \leqq \zeta(\sigma)^n \quad \text{for } \sigma > 1$$

and

(31) $$\zeta(1+\eta) \leqq 1 + \eta^{-1} \quad \text{for } \eta > 0.$$

DEFINITION 1. Given formal Dirichlet series over k

$$f_j(s) = \sum_n \alpha_j(n)|n|^{-s}, \qquad 1 \leqq j \leqq r,$$

we define their scalar product by

$$(f_1 * \cdots * f_r)(s) = \sum_n |n|^{-s} \prod_{j=1}^{r} \alpha_j(n).$$

DEFINITION 2. Let $f(s)=\Sigma_n \alpha(n)|n|^{-s}$ and $g(s)=\Sigma_n \beta(n)|n|^{-s}$. If $|\alpha(n)| \leqq \beta(n)$ for each integral ideal n of k, we write

$$f(s) \ll g(s).$$

LEMMA 2. *We have*

$$(32) \qquad L(s,\rho) \ll \zeta_{k_1}(s) * \cdots * \zeta_{k_r}(s) \ll \zeta_k(s)^{d_1} * \cdots * \zeta_k(s)^{d_r}.$$

PROOF. It follows from the definitions.

LEMMA 3. *Suppose that ρ is of AW type and let $0<\eta<\frac{1}{2}$. In notations of Theorem 1, the following estimate holds:*

$$(33) \qquad |L(s,\rho)| \leqq 5^m \left|\frac{1+s}{1-s}\right|^l \zeta(1+\eta)^{nd}((2+|t|)^{m/2}a(\chi)b(\chi))^{1+\eta-\sigma},$$

where $s=\sigma+it$, $t \in R$, $\sigma \geqq -\eta$.

PROOF. It follows from the functional equation for $L(s,\rho)$ and convexity theorems of Phragmen–Lindelöf type (cf. [20], where a detailed proof of (30) has been given when $\rho \in \mathrm{gr}(k)$).

LEMMA 4. *Suppose that the Dirichlet series*

$$f(s)=\sum_{n=1}^{\infty} a_n n^{-s}$$

converges absolutely for Re $s>1$ *and satisfies two conditions:*

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} = O((\sigma-1)^{-\alpha}) \quad \textit{for } \sigma>1,$$

with $\alpha>0$ independent of σ, and there is a monotonely non-decreasing function $b: R_+ \to R_+$ such that

$$|a_n| \leqq b(n) \quad \textit{for each } n.$$

If x lies in the interval $N+\frac{1}{4}<x<N+\frac{1}{2}$, $N>0$, $N \in \mathbf{Z}$ and $c>1$, then

$$(34) \qquad \sum_{n<x} a_n = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(w)\frac{x^w}{w}\,dw + O\left(\frac{x^c}{T(c-1)^\alpha}\right) + O\left(\frac{b(2x)x\log x}{T}\right).$$

PROOF. It is well known (see, e.g., [24], pp. 53–55, lemma 3.12).

Let $\zeta_{\mathbf{k}}(s)=\zeta_k(s)^{d_1} * \cdots * \zeta_k(s)^{d_r}$. The following lemma is elementary.

LEMMA 5. *There is a sequence of polynomials* $\{h_p(t) \mid p \in S_0\}$ *such that* $h_p(t) \in \mathbf{C}[t]$, $h_p(t) \equiv 1 \pmod{t^2}$, *the degree of* h_p *is not higher than* $d-1$ *and*

$$\zeta_{\mathbf{k}}(s) = \zeta_k(s)^d \prod_{p \in S_0} h_p(|p|^{-s}) \quad \text{for } \operatorname{Re} s > \tfrac{1}{2}. \tag{35}$$

PROOF. It is a special case of Theorem 8 with $\rho_j = d_j I$, where I denotes the identical representation of $W(k)$.

LEMMA 6. *There is an effectively computable constant* $C_7(\varepsilon)$ *such that*

$$\zeta_{\mathbf{k}}(s) \ll \sum_{n=1}^{\infty} (C_7(\varepsilon) n^{\varepsilon}) n^{-s} \quad \text{for each } \varepsilon > 0. \tag{36}$$

PROOF. It follows from the definitions.

REMARK 3. The function $C_7(\varepsilon)$ depends, of course, on the sequence of the fields $k_1, \ldots, k_r$.

Theorem 1 and Theorem 2 follow from Lemmas 1–6 and Cauchy's theorem on residues. To prove (12) let $f(w) = L(w, \rho)$ in (34); take $c = 1 + (\log x)^{-1}$ and apply Cauchy's theorem to the contour of integration consisting of the lines:

$$\{s \mid \operatorname{Re} s = c, \ |\operatorname{Im} s| \leqq T\}, \quad \{s \mid \operatorname{Re} s = (\log x)^{-1}, \ |\operatorname{Im} s| \leqq T\},$$

$$\{s \mid (\log x)^{-1} \leqq \operatorname{Re} s \leqq c, \ \operatorname{Im} s = \pm T\}.$$

Calculating the residue at $s = 1$ and making use of (33) and (32), (36) with $r = 1$, $k_1 = k$ one obtains (12) when T is chosen to be equal to $x^{2/(m+2)}$. To prove (13) one moves the contour of integration to the line $\operatorname{Re} s = \frac{1}{2} + (\log x)^{-1}$ and takes again $c = 1 + (\log x)^{-1}$ in (34) with $f(w) = L(w, \mathfrak{p})$. The estimate (13) follows then from (27), (32), (33), (35) and (36) when we let $T = x^{2/(m+4)}$.

§3. Zero free region for a Hecke function and proof of Theorem 4

For grossencharacters estimate (33) takes the form:

$$|L(s, \chi)| \leqq 5^n \left| \frac{1+s}{1-s} \right|^l \zeta(1+\eta)^n ((2 + |t|)^{n/2} a(\chi) b(\chi))^{1+\eta-\sigma}, \tag{37}$$

where $\chi \in \operatorname{gr}(k)$,

$$l = \begin{cases} 1, & \chi = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$\sigma \geqq -\eta$, $s = \sigma + it$, $t \in \mathbf{R}$, $0 < \eta < \frac{1}{2}$. Classical reasoning (cf., e.g., [19, Ch. 7]) leads now to the following proposition.

PROPOSITION 1. *There are c_5 and c_6 such that $L(s,\chi) \neq 0$ in the region*

$$\operatorname{Re} s > \begin{cases} 1 - (c_5 \log(a(\chi)b(\chi)(2 + |t|)^{n/2}))^{-1}, & |t| > c_6 \\ 1 - (c_5 \log(a(\chi)b(\chi)(2 + c_6)^{n/2}))^{-1}, & |t| \leqq c_6 \end{cases} \tag{38}$$

save for a possible exceptional zero, when $\chi^2 = 1$, which must be real and simple. Here $c_5 > 0$, $c_6 > 0$, $t := \operatorname{Im} s$.

PROOF. If $\chi^2 = 1$, then χ is a character of finite order and the assertion follows from Lemma 2.3 in [6, p. 277]. Suppose that $\chi^2 \neq 1$ and let $L(\alpha + it_0, \chi) = 0$, $t_0 \in R$, $\alpha \geqq \frac{3}{4} + \beta$, $0 < \beta \leqq \frac{1}{4}$.

Let $s_0 = 1 + \beta + it_0$, $s_1 = 1 + \beta + 2it_0$. One writes

$$3 \operatorname{Re} \frac{\zeta_k'}{\zeta_k}(1 + \beta) + 4 \operatorname{Re} \frac{L'}{L}(s_0, \chi) + \operatorname{Re} \frac{L'}{L}(s_1, \chi^2) \leqq 0. \tag{39}$$

A classical argument making use of (37) and a function theoretical lemma of E. Landau (cf., e.g., [19, p. 384], Satz 4.4 and Satz 4.5) allows one to estimate the three terms in (39) from below. These estimates when substituted in (39) lead to the assertion of Proposition 1.

Let $\psi(t,\chi) = c_5^{-1}[\log(a(\chi)b(\chi)(2 + |t|)^{n/2})]^{-1}$. Choose c_6 in such a way that $c_6 > \psi(c_6)$ and the circle

$$\{s \mid s \in \mathbf{C},\ |s - (1 + \tfrac{1}{2}\psi(t) + it)| < \psi(t)\}, \qquad t := \operatorname{Im} s,$$

is contained in the region (38) when $|t| \leqq c_6$. A simple calculation making use of a classical function theoretical lemma (see, e.g., Satz 4.3 in [19, p. 383]) allows one to deduce from Proposition 1 and (37) the following statement.

LEMMA 7. *There is c_7 in the interval $0 < c_7 < 1$ such that the function*

$$f(s) = \log\left(L(s,\chi)\left(\frac{s-\alpha}{s}\right)^{\nu_1(\chi)}\left(\frac{s-1}{s}\right)^{\nu_2(\chi)}\right),$$

where

$$\nu_2(\chi) = \begin{cases} 0, & \chi \neq 1 \\ 1, & \chi = 1 \end{cases}$$

and

$$\nu_1(\chi) = \begin{cases} -1 & \textit{when } L(\alpha, \chi) = 0 \textit{ and } 1 - c_7\psi(c_6, \chi) \leqq \alpha \leqq 1, \\ 0 & \textit{when there is no such } \alpha, \end{cases}$$

is regular in the region

$$\text{(40)} \qquad \operatorname{Re} s \geqq \begin{cases} 1 - c_7\psi(t, \chi) & \textit{when } |t| > c_6, \\ 1 - c_7\psi(c_6, \chi) & \textit{when } |t| \leqq c_6, \end{cases} \qquad t = \operatorname{Im} s.$$

Moreover, in the region defined by (40) *the following estimates hold*:

$$\text{(41)} \qquad f(s) = O(\psi(t, \chi)^{-1}), \quad \frac{f'}{f}(s) = O(\psi(t, \chi)^{-2}).$$

If $|t| > c_6$ *and* $\operatorname{Re} s \geqq 1 - c_7\psi(t, \chi)$, *then*

$$\text{(42)} \quad f(s) = O(n \log(5(1 + \eta^{-1}) + (\eta + c_7\psi(t, \chi))\log(a(\chi)b(\chi)(2 + |t|)^{n/2})))$$

for each η *in the interval* $0 < \eta < \frac{1}{2}$.

Theorem 4 is a formal consequence of Proposition 1, Lemma 7 and identity (9). Suppose first that $\chi \in \operatorname{gr}(k)$. Applying Lemma 4 to the function

$$s \mapsto \frac{L'}{L}(s, \chi)$$

one deduces from Proposition 1 and (41) an estimate

$$\text{(43)} \qquad \begin{aligned} \sum_{|p| < x} \chi(p) = \nu_2(\chi) \int_2^x \frac{du}{\log u} \\ + O(x^{\alpha}) + O\left(x \exp\left(-c_8 \frac{\log x}{\log(a(\chi)b(\chi)) + \sqrt{n} \log x}\right)\right), \end{aligned}$$

where $c_8 > 0$ and α denotes the possible exceptional zero of $L(s, \chi)$ when

$\chi^2 = 1$; here p varies over prime divisors of k. Taking the logarithmic derivative in (9) one obtains, after an easy calculation, an estimate

$$\sum_{|p|<x} \chi(p) = \sum_{j=1}^{\nu} e_j \sum_{|\mathfrak{p}|<x} \psi_j(\mathfrak{p}) + O(m\sqrt{x}) \tag{44}$$

with m defined by (11); here $\mathfrak{p}$ varies over prime divisors of E_j. Relations (16) and (17) follow from (44) when one applies (43) to each of the sums $\Sigma_{|\mathfrak{p}|<x} \psi_j(\mathfrak{p})$, $1 \leqq j \leqq \nu$. This proves Theorem 4.

§4. Proof of Theorem 3

We return to notations of Theorem 1 and observe that (42) and (9) imply, after adjusting η, the following assertion.

LEMMA 8. *There are c_{13} and c_9 such that*

$$|L(s,\rho)| \leqq (m+1)^{c_{13}m} a(\chi) b(\chi)(2+|t|)^{1/2} \tag{45}$$

for

$$\operatorname{Re} s \geqq 1 - \left(\frac{c_7}{c_5}\right) \frac{1}{\log(a(\chi)b(\chi)(1+|t|)^m)}, \qquad |t| \geqq c_9; \quad t := \operatorname{Im} s.$$

We make now a few remarks concerning summatory properties of the coefficients of Dirichlet series representing a meromorphic function in the region of the shape (38). Let $f(s)$ be a function meromorphic in the region

$$\mathcal{B} = \left\{ s = u + it \;\middle|\; u \geqq 1 - \frac{c_{10}}{\log(b_1(2+|t|)^m)},\, t \in \mathbf{R} \right\},$$

where $b_1 \geqq 1$, $c_{10} \geqq 1$, and suppose that the following conditions hold:

(i) for $\operatorname{Re} s > 1$ this function is given by an absolutely convergent Dirichlet series:

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad \operatorname{Re} s > 1; \tag{46}$$

(ii) there is c_9 such that for $s \in \mathcal{B}$ and $|t| \geqq c_9$, $t := \operatorname{Im} s$, we have an estimate

$$f(s) = O(b_2(2+|t|)^{\gamma}) \quad \text{with } 0 < \gamma < 1,\, b_2 \geqq 1; \tag{47}$$

(iii) the function $f(s)$ has no singularities in $\mathcal{B}$ save for a finite number of real poles in the interior of $\mathscr{B}$; let $g_i \geqq 1$ be the multiplicity of the pole α_i of $f(s)$, $\alpha_i \in \mathscr{B}$. Let, for $x > 1$, $x \notin Z$,

$$A(x) = \sum_{n<x} a_n.$$

PROPOSITION 2. *Assume* (i)–(iii). *Then the following estimate holds*:

$$A(x) = \sum_i x^{\alpha_i} P_i(\log x) + R(x), \tag{48}$$

where P_i is a polynomial of degree $g_i - 1$ exactly computable in terms of $f^{(\nu)}(\alpha_i)$, $0 \leqq \nu \leqq g_i - 1$; in particular, P_i coincides with the residue of $f(s)$ at $s = \alpha_i$ when $g_i = 1$. Moreover, there is $c_{11} > 0$ such that

$$R(x) = O(b_1 b_2 x \exp(c_{11} m^{-1} \sqrt{\log x})) + \left(\sum_{x \leqq n < x\beta} |a_n| \right) \tag{49}$$

where $\beta = 1 + \exp(-c_{11} m^{-1} \sqrt{\log x})$. If $a_n \geqq 0$ for each n, then

$$R(x) = O(b_1 b_2 x \exp(-c_{11} m^{-1} \sqrt{\log x})). \tag{50}$$

PROOF. Let

$$A_1(x) = \sum_{n<x} a_n \log(xn^{-1}).$$

In view of (46) and (47), one obtains from the identity

$$A_1(x) \equiv \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s^2} f(s) ds, \qquad c > 1, \quad x > 1, \quad x \notin Z,$$

an estimate

$$A_1(x) = \sum_i x^{\alpha_i} \tilde{P}_i(\log x) + O\left(b_1 b_2 x \exp\left(-\frac{(1-\gamma)}{m} c_{12} \sqrt{\log x} \right) \right) \tag{51}$$

with $c_{12} > 0$ and $\tilde{P}_i$ satisfying the conditions mentioned in Proposition 2.

Let $1 \leqq \beta \leqq 2$. Then

$$A(x) = (A_1(\beta x) - A_1(x))(\log \beta)^{-1} + O\left(\sum_{x \leqq n < x\beta} |a_n| \right). \tag{52}$$

Estimates (51) and (52) imply (49). If $a_n \geqq 0$ for each n, then

$$A_1(x) - A_1(x\beta^{-1}) \leqq A(x) \log \beta \leqq A_1(x\beta) - A_1(x). \tag{53}$$

Estimate (50) follows from (51) and (53). This proves the Proposition.

LEMMA 9. *Write* $\zeta_k(s) = \sum_{l=1}^{\infty} a_l l^{-s}$. *We have*

$$\sum_{l<x} a_l = xP(\log x) + O(C_8(\varepsilon)|D|^{d/2}x^{1-2/(4+n)+\varepsilon}), \tag{54}$$

where D denotes the discriminant of k and P is a polynomial of degree $d-1$.

PROOF. It follows from Theorem 2 and Lemma 5.

COROLLARY 1. *Let* $1 \leqq \beta \leqq 2$. *We have*

$$\sum_{x \leqq |n| < x\beta} |a(n, \rho)| = O((\beta - 1)x(\log x)^{d-1}C(k, d)) \tag{55}$$

with an exactly computable $C(k, d)$ *depending on k and d only.*

PROOF. It follows from (54) and (32).

In view of Lemma 1 and Lemma 8 the function

$$f: s \mapsto L(s, \boldsymbol{\rho})$$

satisfies conditions (i)–(iii). Therefore the assertion of Theorem 3 follows from (48), (49) and (55).

§5. Conclusion of the proofs

Let us return to notations of Theorems 5 and 6, so that

$$\rho_j = \mathrm{Ind}_{W(k_j)}^{W(k)} \chi_j, \qquad \chi_j \in \mathrm{gr}(k_j), \quad \rho = \rho_1 \otimes \cdots \otimes \rho_r. \tag{56}$$

Being a product of monomial representations, ρ can be decomposed in a direct sum of monomial representations (cf. [11]), so that (9) holds with $e_j = 1$ for each j (cf. [13, Proposition 2]). In particular, ρ is of AW type. Therefore (19) follows from (13). Moreover, in this case one can exactly compute ψ_j, $1 \leqq j \leqq r$ (cf. [16, §II.5]); since the polynomial $P_{\boldsymbol{\rho}}(t)$ is obtained by computing the residue of the function

$$s \mapsto \frac{x^s}{s} L(s, \rho),$$

its shape is determined by (25) and (9) as soon as ψ_j are known. This completes the proof of Theorem 5. To prove (20) let

$$B(\chi, x) = \sum_{|p|<x} a(p, \mathfrak{p}),$$

where p ranges over prime divisors of k and $a(p, \mathfrak{p})$ is defined as in (1) with ρ_j, $1 \leqq j \leqq r$, given by (56). Obviously,

$$\sum_{|\mathfrak{p}|<x} \chi(\mathfrak{p}) = B(\chi, x) + \sum_{|p|<x^{1/2}} \sum_{m=2}^{d} \sum_{N_{\mathbf{k}/k}\mathfrak{p}=p^m} \chi(\mathfrak{p}), \tag{57}$$

where $\mathfrak{p}$ ranges over $J_0(\mathbf{k})$ and we let

$$N_{\mathbf{k}/k}\mathfrak{a} = N_{k_1/k}\mathfrak{a}_1 \quad \text{for } \mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r) \quad \text{in } J(k).$$

Identity (57) implies an estimate

$$\sum_{|\mathfrak{p}|<x} \chi(\mathfrak{p}) = B(\chi, x) + O(d\sqrt{x}). \tag{58}$$

On the other hand, one observes from (25) that in $C[[t]]$ the following identity holds:

$$l_p(t, \chi) = \det(I - \rho(\sigma_p)t)^{-1}\Phi_p(t)l_p(t), \tag{59}$$

where $l_p(|p|^{-s}, \chi)$ denotes the local factor of $L(s, \mathfrak{p})$, so that

$$l_p(t, \chi) = \sum_{l=0}^{\infty} a(p^l, \mathfrak{p})t^l. \tag{60}$$

Since $l_p(t) = 1$ for $p \notin S_0(\mathfrak{p})$ and $\Phi_p(t) \equiv 1 \pmod{t^2}$, we deduce from (59) and (60) the following relations:

$$a(p, \chi) = \chi(p) \quad \text{for } p \notin S_0(\mathfrak{p}), \tag{61}$$

and

$$a(p, \chi) = \chi(p) + O(d) \quad \text{for each } p. \tag{62}$$

Estimate (20) with $C_4(\chi) = d\,|S_0(\mathfrak{p})|$, where $|S_0(\mathfrak{p})|$ denotes the cardinality of $S_0(\mathfrak{p})$, follows from (58), (60) and (61). This proves Theorem 6.

We turn now to the proof of conditional estimates (22) and (23). While (23) requires the full strength of the Riemann Hypothesis (21), estimate (22) follows from the (generalised) Lindelöf conjecture: for $\operatorname{Re} s > \frac{1}{2} + \sigma_1$ with fixed positive σ_1, we have

$$L(s, \rho)^{\alpha} = O_{\varepsilon}\left(\left|\frac{s+1}{s-1}\right|^{\alpha l} (15^m a(\chi) b(\chi)(2 + |t|)^{m/2})^{m\varepsilon}\right), \tag{63}$$

where $\alpha \in \{-1, 1\}$, $\varepsilon > 0$, $t = \operatorname{Im} s$, while m and l have the same meaning as in (33). A classical argument, [10] (cf. [24, §14.2]), shows that (63) follows from the Generalised Riemann Hypothesis (21); alternatively, one can deduce (63) from two assumptions (cf. [18]):

(64.1) $$L(s, \rho) \neq 0 \quad \text{for } \operatorname{Re} s > \tfrac{1}{2},$$

and ρ is of AW type; moreover, the second assumption may be weakened to:

(64.2) $$L(s, \rho)\left(\frac{s+1}{s-1}\right)^{l} \quad \text{is holomorphic for } \operatorname{Re} s > \tfrac{1}{2}.$$

Obviously, (21) implies (64).

LEMMA 10. *Suppose that $f(s)$ is holomorphic for* $\operatorname{Re} s > \frac{1}{2}$ *and the following two estimates hold*:

(65.1) $$|f(s)| < B(f)(2 + |t|)^{m} \quad \textit{for } \operatorname{Re} s > \tfrac{1}{2},$$

and

(65.2) $$\log f(s) = O(m \log(2 + \eta^{-1})) \qquad \textit{for } \operatorname{Re} s > 1 + \eta, \quad \eta > 0,$$

where $t = \operatorname{Im} s$, $m > 0$, $B(f) > 1$. *Then*

(66) $$\log f(s) = O_{\varepsilon}(m[\log(B(f)(2 + |t|)^{m})]^{2(1-\sigma)+\varepsilon})$$

whenever $\frac{1}{2} < \sigma \leqq 1$, $\sigma := \operatorname{Re} s$; $\varepsilon > 0$.

PROOF. The well-known argument, [10] (or [24, §14.2]), shows that (66) follows from (65).

COROLLARY 2. *In notations and under conditions of Lemma* 10, *we have*

(67) $$f(s)^{\alpha} = O_{\varepsilon}((B(f)(2 + |t|)^{m})^{m\varepsilon}), \qquad \varepsilon > 0,$$

where $\alpha \in \{-1, 1\}$, $\sigma \geqq \frac{1}{2} + \sigma_1$, σ_1 *is a fixed positive real number.*

Assume that (64) holds. Let

$$f(s) = L(s, \rho)\left(\frac{s-1}{s+1}\right)^{l}$$

and let

$$B(f) = 15^{m}\alpha(\chi)b(\chi).$$

In view of (3) and (33), the function $f(s)$ satisfies conditions of Lemma 10. Therefore (67) holds, and we obtain (63). To prove (22) one remarks that conditions (64) and Theorem 8 allow one to move the contour of integration in (34), with $f(s) = L(s, \mathfrak{p})$, to the line $\operatorname{Re} s = \frac{1}{2} + \sigma_0$ for any positive σ_0. Estimate (22) follows then from (63), (27) and (32). Since

$$\sum_{|\mathfrak{a}|<x} \chi(\mathfrak{a}) = A(x, \mathfrak{p})$$

with $\mathfrak{p}$ defined by (56), the Generalised Riemann hypothesis (or (64)) implies an estimate of the shape (22) for this sum. Moreover, as it has been already remarked, the polynomial $P_{\mathfrak{p}}(t)$ can be precisely evaluated in this case.

Lemma 11. *Let f be an entire function satisfying the following condition:*

$$|f(\sigma + it)| < \varphi_f(|t|) \quad \textit{for } -\tfrac{1}{4} \leqq \sigma \leqq 5, \quad t \in \mathbf{R}, \tag{68}$$

where $\varphi_f : \mathbf{R} \to \mathbf{R}$ *is a non-decrasing function and* $\varphi_f(u) \geqq 1$ *for* $u \geqq 0$. *Let* $N(f, T)$ *denote the number of zeros of* $f(s)$ *in the rectangle* $0 \leqq \operatorname{Re} s \leqq 1$, $|\operatorname{Im} s| \leqq T$; $T > 0$. *Then*

$$N(f, T+1) = N(f, T) + O(\log \varphi_f(T+3)). \tag{69}$$

The proof of this lemma is completely analogous to the proof of Theorem 9.2 in [24, p. 178] and may be omitted. One applies this lemma to estimate the number of zeros $N(\chi, T)$ of the function $s \mapsto L(s, \chi)$, $\chi \in \operatorname{gr}(k)$, in the rectangle $0 \leqq \operatorname{Re} s \leqq 1$, $|\operatorname{Im} s| \leqq T$.

Let

$$A_\chi = 5^n \zeta(\tfrac{5}{4}) a(\chi) b(\chi).$$

Estimates (33) and (69) with $f(s) = L(s, \chi)(1-s)^{g(\chi)}$, where $g(\chi) = 1$ when $\chi = 1$ and $g(\chi) = 0$ when $\chi \neq 1$, give

$$N(\chi, T+1) = N(\chi, T) + O(\log(A_\chi (2+T)^n)). \tag{70}$$

On the other hand, a classical argument (cf., e.g., [19, Ch. VII §4]) leads to an exact formula:

$$\begin{aligned} &\sum_{|p|^m < x} \chi(p^m) \log |p| \\ &= g(\chi) x + \sum_\alpha{}' \frac{x^\alpha}{\alpha} + O\left(\frac{x[n(\log x)^2 + \log^2(A_\chi(2+T)^n)]}{T} \right) \end{aligned} \tag{71}$$

where p ranges over prime divisors in k and m ranges over the natural integers subject to the condition $|p|^m < x$; in the right hand side α ranges over the zeros of $L(s,\chi)$ in the critical strip $0 \leqq \operatorname{Re} s \leqq 1$. The proof of this exact formula makes use of the functional equation for $L(s, \chi)$ and estimate (70). The (generalised) RiemannHypothesis gives, in view of (70),

$$\sum_{\alpha}{}' \frac{x^\alpha}{\alpha} = O(x^{1/2}(\log T)(\log(A_\chi(2+T)^n))). \tag{72}$$

By partial summation in the left hand side of (71), taking $T = x$ one deduces from (71) and (72) an estimate

$$\sum_{|p|<x} \chi(p) = g(\chi) \int_2^x \frac{du}{\log u} + O(x^{1/2}(\log A_\chi + n \log x)), \tag{73}$$

where $\chi \in \operatorname{gr}(k)$. To prove (23) it is enough to apply (73) to each of the sums $\Sigma_{|\mathfrak{p}|<x} \psi_j(\mathfrak{p})$, $1 \leqq j \leqq \nu$, in (44).

§6. Final remarks and acknowledgements

It is known classically that

$$\sum_{|\mathfrak{a}|<x} \chi(\mathfrak{a}) = g(\chi)\omega_k x + O(x^{(n-1)/(n+1)}) \tag{74}$$

for $\chi \in \operatorname{gr}(k)$, where ω_k denotes the residue of $\zeta_k(s)$ at $s = 1$; moreover, if

$$\sum_{|\mathfrak{a}|<x} \chi(\mathfrak{a}) = g(\chi)\omega_k x + O(x^\gamma) \tag{75}$$

with $\chi \in \operatorname{gr}(k)$, then $\gamma > (n-1)/2n$ for $n > 1$ (cf. [8], [2]). Imitating the argument of E. Landau, [8], one should be able to obtain an estimate

$$\sum_{|n|<x} a(n,\chi) = xP_\rho(\log x) + O(x^{1-2/(1+m)}) \tag{76}$$

for representations ρ of AW type, slightly improving on (12); however, in (74)–(76) the constants implied by O symbols depend on χ in a non-specified way. It is tempting to conjecture that actually if ρ is of AW type, then

$$\sum_{|n|<x} a(n,\chi) = xP_\rho(\log x) + O(x^\gamma) \tag{77}$$

with $\gamma < \frac{1}{2}$ (in view of Ω-theorem (75), we have $\gamma \geqq \frac{1}{2} - 1/2m$ when $\rho \in \operatorname{gr}(k)$). Conversely, (77) with $\gamma < \frac{1}{2}$ implies the holomorphy of $L(s,\rho)$ in $\mathbf{C} \setminus \{1\}$. In this context Professor P. Deligne has asked me about the error term in the estimate for $\Sigma_{|n|<x} a(n,\chi)$ when one doesn't know whether $L(s,\rho)$ is holomorphic or not. To answer this question we have written a short paper, [14] (cf. also [15, Ch. II §4]), where Theorem 3 has been proved in a form slightly less precise than here. Subsequent consultation with Professor P. Deligne and Professor O. Gabber allowed us to simplify the argument; these simplifications are incorporated in the proof of Theorem 3 given here. Considerations leading to Proposition 2 are of classical origin and analogous statements can be found in the literature (cf., e.g., [4, §2] and [7, §242, Satz 62]). The author is indebted to Professor H. Delange for a private communication containing an alternative proof of an estimate leading to (48) and (50) and to Professor E.-U. Gekeler who has pointed out that (77) with $\gamma < \frac{1}{2}$ would imply the Artin–Weil conjecture for $L(s,\rho)$.

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